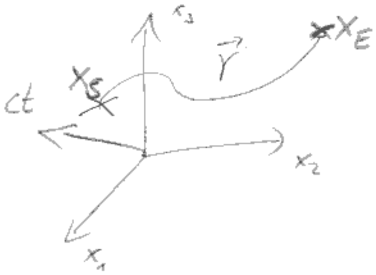
Invariance of the D'Alembertoperator under Lorentztransformations

$[g_{ij} = \eta_{ij}]$

$\boxed{1}$ Definition: The length of a curve $\vec{\gamma}$.

As a parametrisation we can choose for example t.



$$L(\vec{\gamma}) = \int_a^b \|\dot{\vec{\gamma}}\| \, dt .$$

If our space has a symmetric, non degenerated bilinear form, we can write $\|\dot{\vec{\gamma}}\| = \sqrt{g(v,v)}$.

$X(a) = X_S$

$X(b) = X_E$

$$= \int_a^b \sqrt{g_{ij}\, \dot{x}_i \dot{x}_j} \, dt \qquad \dot{x}_i = \frac{dx_i}{dt}$$

The so defined length is invariant under change of parametrisation:

$$S = S(t) \rightarrow dS = \frac{\partial S}{\partial t} dt \rightarrow dt = \left|\frac{\partial t}{\partial S}\right| dS$$

$$= \int_c^d \sqrt{g_{ij} \frac{\partial x_i}{\partial S}\frac{\partial S}{\partial t} \cdot \frac{\partial x_j}{\partial S}\frac{\partial S}{\partial t}} \left|\frac{\partial t}{\partial S}\right| dS$$

$X(S(a)) = X(c) = X_S$

$X(S(b)) = X(d) = X_E$

$$= \int_c^d \sqrt{g_{ij} \frac{\partial x_i}{\partial S}\frac{\partial x_j}{\partial S}} \cdot \cancel{\left|\frac{\partial S}{\partial t}\right|} \cancel{\left|\frac{\partial t}{\partial S}\right|} dS = \int_c^d \sqrt{g_{ij} \dot{X}_i \dot{X}_j} \, dS$$

Canonical (Natural) Parametrisation:

$$c\tau = \int_0^\tau \sqrt{g_{ij} \dot{x}_i \dot{x}_j} \, dt$$

It is obvious that a such a parametrisation exists. Consider $\tau = \int_0 \sqrt{g_{ij}\dot{x}_i\dot{x}_j}\, dt'$. Say we drive on the curve a little bit slower, then a is going to be larger then τ. If we drive to fast a is going to be bigger. The whole thing is smooth

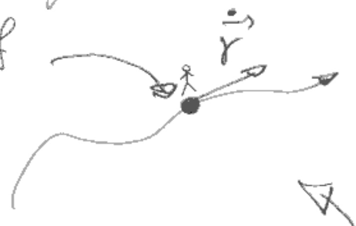
$\rightarrow \exists$ parametrisation s.t. $\tau = \int_0^\tau \sqrt{g_{ij}\dot{x}_i\dot{x}_j}\, dt$

$$c\tau = \int_0^t (c^2 - v^2)^{\frac{1}{2}} \, dt' \quad (2)$$

$\rightarrow$ for $v = \text{const}$ $\quad c\tau = (c^2 - v^2)^{\frac{1}{2}} t \quad (1)$

$\rightarrow \quad c\tau = c\int_0^\tau d\tau \;\hat{=}\;$ Coordinate system where $v = 0$!

Hence we have seen that $c\tau$, the length of the curve, is the time passed in the rest frame of



and we have the relation

$$\tau = (1-\beta^2)^{\frac{1}{2}} \cdot t \qquad \beta = \frac{v}{c}$$

see ①. Define $\gamma = \frac{1}{\sqrt{1-\beta^2}} \rightarrow \boxed{t = \gamma \cdot \tau} \quad (1)$

(1) Proofs that what we have said here, is indeed the case, because (1) is experimentally verifiable.

Go back to ①. We see that the canonical parametrisation for constant v results into a coordinatesystem where $v = v$.

Go back to (2) and integrate for $v = $ const.

$$c\tau = (c^2 - v^2)^{\frac{1}{2}} \cdot t \rightarrow c^2\tau^2 = (c^2 - v^2) \cdot t^2$$

$$\rightarrow c^2 \cdot \tau^2 = c^2 t^2 - v_1^2 t^2 - v_2^2 t^2 - v_3^2 t^2$$

Hence $x^\mu \cdot x_\mu$ is $c^2 \cdot$ „rest frame time". $\equiv x^\mu \cdot x_\mu = x^\mu \eta_{\mu\nu} x^\nu$

τ has to be the same „seen" from any inertial frame. (Inertialsystem).

→ A transformation leading from one inertial frame to the other, has to leave τ untouched, that means

$$c^2\tau^2 = x^\mu x_\mu = x^\mu \eta_{\mu\nu} x^\nu \overset{!}{=} (\Lambda^\alpha{}_\beta x^\beta)\, \eta_{\alpha\beta}\, (\Lambda^\beta{}_\gamma x^\gamma)$$

$$= \Lambda^\alpha{}_\beta\, \eta_{\alpha\beta}\, \Lambda^\beta{}_\gamma \cdot x^\beta \cdot x^\gamma$$

$$\rightarrow \boxed{\Lambda^\alpha{}_\beta\, \eta_{\alpha\beta}\, \Lambda^\beta{}_\gamma = \eta_{\beta\gamma}}$$

Where we notated the transformation as $\Lambda^{\alpha}{}_{\beta}$.

$$\boxed{\Lambda^T \eta \Lambda = \eta} \qquad (3)$$

All transformations obeying (3) are called Lorentz transformations.

Now we can show the invariance of $\Box$.

$$\Box = \partial_\mu \partial^\mu = \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \cdot \frac{\partial}{\partial x^\nu}$$

$$\frac{\partial}{\partial x^\mu} f = \frac{\partial f}{\partial y^\nu} \frac{\partial y^\nu}{\partial x^\mu} \qquad \frac{\partial}{\partial x^\mu} \longrightarrow \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu}$$

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial}{\partial x^\nu} f\right) = \frac{\partial}{\partial x^\mu}\left(\frac{\partial f}{\partial y^\alpha} \frac{\partial y^\alpha}{\partial x^\nu}\right) = \left(\frac{\partial}{\partial x^\mu}\left(\frac{\partial f}{\partial y^\alpha}\right)\right)\frac{\partial y^\alpha}{\partial x^\nu} + \left(\frac{\partial f}{\partial y^\alpha}\right) \cdot \underbrace{\left(\frac{\partial}{\partial x^\mu} \frac{\partial y^\alpha}{\partial x^\nu}\right)}_{=0}$$

because Transformations (Lorentz) are linear.

$$\longrightarrow \quad \eta^{\mu\nu} \frac{\partial y^\beta}{\partial x^\mu} \frac{\partial y^\alpha}{\partial x^\nu} \cdot \frac{\partial f}{\partial y^\beta \partial y^\alpha}$$

$$= \eta^{\mu\nu} \Lambda^\beta{}_\mu \Lambda^\alpha{}_\nu \frac{\partial}{\partial y^\beta \partial y^\alpha}$$

$$\underset{\uparrow}{=} \eta^{\alpha\beta} \frac{\partial}{\partial y^\beta \partial y^\alpha} = \partial_\alpha \partial^\alpha$$

Definition Lorentztransformations (3)

$\longrightarrow$ $\Box$ is invariant under Lorentz transformations.

$\boxtimes$